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A BRIEF ACCOUNT OF THE ESSENTIAL FEATURES OF GRASSMANN'S EXTENSIVE ALGEBRA.

[Given by the author in Grunert's Archiv, Vol. VI, 1845.]

TRANSLATED BY PROF. W. W. BEMAN, ANN ARBOR, MICH.

[Continued from page 97.]

6. *If all the elements of a field of the* n*th order be subjected to one and the same mode of variation, leading to new elements (not contained in that field), then the aggregate of the elements producible by this mode of variation and its opposite is called a field of the* (n+1)*th order; the field of the third order corresponds to the plane, that of the fourth to space in general.*

If the points of a right line all move in one and the same direction, leading to new points (not contained in that right line), then is the aggregate of the points producible by this motion and its opposite, the plane, and if we proceed in the same way with the points of the plane, we get all space. If we substitute here for the spatial notions the abstract ones given above, and keep the transition from one order to the next higher general, the idea just mentioned is obtained.

II. BEARING OF THE CALCULUS EMPLOYED IN MY EXTENSIVE ALGEBRA UPON GEOMETRY EXPLAINED.

7. *In my extensive algebra there appears a peculiar calculus, which, transferred to geometry, is of inexhaustible fertility, and here (in geometry) consists in subjecting spatial figures (points, lines, etc.) immediately to calculation.*

For example, the right line drawn through two points, is, with respect to its position and magnitude, regarded as the join (Verknüpfung) of those p'ts, and, indeed, as a peculiar kind of multiplication (see No. 15 below); likewise the triangle included by three points, with respect to its area and the position of its plane, as the product of three points, so that this product is zero, when the area of the triangle is zero, i. e., when the three points lie in a right line; further, in a sense to be explained more fully later (see No. 22 and Prob. 18), the point of intersection of two right lines is regarded as their product.

8. *The effect of the application of this calculus to geometry is to unite the synthetic and analytic methods,* i. e., *to transplant the advantages of each into the soil of the other, while side by side with every construction we have a simple analytical operation, and conversely.*

For illustration, take the following example. As is well known, the vertex γ of a variable triangle, whose other two vertices α, β move in fixed right lines A and B, and whose sides pass through three fixed points a, b, c, describes a conic section. If a, b, c, are the fixed points through which pass the sides opposite the vertices α, β, γ, respectively, then we see (No. 7) that γaB represents the vertex β, $\gamma aBcA$, the vertex α, and, since the points α, b, γ, lie in one right line, their product being zero, we have the equation

$$\gamma aBcAb\gamma = 0$$

as the equation of a conic section described by γ. We see that this equation is of the second degree with respect to γ, and in this we already have a presentiment of an important law applicable to all algebraic curves.

III. SIMPLEST RULES OF OPERATION FOR THE NEW ANALYSIS.

The combinations which occur in this part of the extensive algebra are addition, subtraction, combinatory multiplication, combinatory division.

9. *For all kinds of addition and subtraction, the ordinary processes hold good.*

10. *For all kinds of multiplication and division, the following law holds good: Instead of multiplying or dividing an aggregate of terms by a signless expression in any way whatever, we can, without changing the final result, multiply or divide the separate terms in the same way* by this expression, and unite the separate products or quotients into an aggregate by placing before each one the sign of that term by whose multiplication or division it was obtained: further a numerical factor associated with any factor of the product may be associated with any other, or with the product: finally* $A \div A$ *is always* 1, *when* A *is not zero.*

11. *A product* a . b . c . . . *I call a combinatory product, when in addition to law No.* 10, *the following law holds good, that if two consecutive factors of the product* a . b . c . . . . *be interchanged, the product takes the opposite value; and I call* a, b, c, . . . *and their sums or differences in that case factors of the first order* (Ordnung).

For example, according to this, $a \,.\, b \,.\, c \,.\, d = - a \,.\, c \,.\, b \,.\, d$.

12. *If in a combinatory product two factors of the first order are equal to each other, the product is zero.*

For example, $a \,.\, b \,.\, b \,.\, d = 0$ (as is seen at once if b and c are made equal in the example of No. 11). The following problems will serve to elucidate this method of multiplication:

Prob. 1. To develop the combinatory product $(\alpha_1\varepsilon_1 + \alpha_2\varepsilon_2 + \alpha_3\varepsilon_3) \,.\, (\beta_1\varepsilon_1 + \beta_2\varepsilon_2 + \beta_3\varepsilon_3) \,.\, (\gamma_1\varepsilon_1 + \gamma_2\varepsilon_2 + \gamma_3\varepsilon_3)$, where $\alpha_1, \alpha_2, \alpha_3$; $\beta_1, \beta_2, \beta_3$; $\gamma_1, \gamma_2, \gamma_3$ represent numerical quantities, and $\varepsilon_1, \varepsilon_2, \varepsilon_3$, combinatory factors of the first order. By applying Rules (9–12) we finally obtain the expression

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*This expression refers not only to the method of combination in general, but also to the position of the factor in the product.

$$(\alpha_1\beta_2\gamma_3-\alpha_1\beta_3\gamma_2+\alpha_3\beta_1\gamma_2-\alpha_3\beta_2\gamma_1+\alpha_2\beta_3\gamma_1-\alpha_2\beta_1\gamma_3)\varepsilon_1.\varepsilon_2.\varepsilon_3.$$

Prob. 2. To solve three equations of the first degree, involving three unknown quantities, by the rules of combinatory multiplication.

Let the three equations be

$$(1)\qquad \begin{cases}\alpha_1 x+\beta_1 y+\gamma_1 z=\delta_1,\\ \alpha_2 x+\beta_2 y+\gamma_2 z=\delta_2,\\ \alpha_3 x+\beta_3 y+\gamma_3 z=\delta_3.\end{cases}$$

Multiply the three equations respectively by three combinatory factors of the first order ε_1, ε_2, ε_3, whose product is not zero, add, and assume

$$(2)\qquad \begin{cases}\alpha_1\varepsilon_1+\alpha_2\varepsilon_2+\alpha_3\varepsilon_3=a,\\ \beta_1\varepsilon_1+\beta_2\varepsilon_2+\beta_3\varepsilon_3=b,\\ \gamma_1\varepsilon_1+\gamma_2\varepsilon_2+\gamma_3\varepsilon_3=c,\\ \delta_1\varepsilon_1+\delta_2\varepsilon_2+\delta_3\varepsilon_3=d;\end{cases}$$

we then get the equation

$$(3)\qquad xa+yb+zc=d.$$

Multiply this equation combinatorily by $b.c$, and we get, since $b.b.c$ and $c.b.c$ are zero (No. 12), the equation

$$x.a.b.c=d.b.c.$$

$$\therefore\ x=\frac{d.b.c}{a.b.c};$$

and in a similar way we find y and z, and get

$$(4)\qquad x=\frac{d.b.c}{a.b.c},\quad y=\frac{a.d.c}{a.b.c},\quad z=\frac{a.b.d}{a.b.c}.$$

These expressions (in which the laws of combinatory multiplication allow no cancellation of the separate combinatory factors) are extremely convenient in analysis. If we wish the unknown quantities expressed in the ordinary form, we have only to substitute from equation (2), to develop according to Prob. 1, and cancel $\varepsilon_1\ \varepsilon_2\ \varepsilon_3$ (No. 10) in numerator and denominator; e. g. we find

$$(5)\quad x=\frac{\delta_1\beta_2\gamma_3-\delta_1\beta_3\gamma_2+\delta_3\beta_1\gamma_2-\delta_3\beta_2\gamma_1+\delta_2\beta_3\gamma_1-\delta_2\beta_1\gamma_3}{\alpha_1\beta_2\gamma_3-\alpha_1\beta_3\gamma_2+\alpha_3\beta_1\gamma_2-\alpha_3\beta_2\gamma_1+\alpha_2\beta_3\gamma_1-\alpha_2\beta_1\gamma_3}.$$

We see not only how this process is in general applicable to n equations of the first degree involving n unknown quantities, but also how we can write out the final result with considerable facility as soon as the n equations are given.

IV. ELEMENTARY NOTIONS OF THE DIFFERENT MAGNITUDES AND MODES OF COMBINATION IN GEOMETRY.

13. *The spatial magnitudes of the first order are simple or multiple points, and right lines of definite length and direction.*

If A and B are two points, I represent the right line from A to B, so far as length and direction are fixed in it but nothing more, by $B—A$; I say therefore that $B—A$ can be equal to $B_1—A_1$ then and then only, when the right lines from A to B and from A_1 to B_1 have the same length and direction.

14. *The spatial magnitudes of the* n*th order arise from combinatory multiplication of* n *magnitudes of the first order* (Stufe), *which are taken as factors of the first order* (Ordnung).

In this case if the factors of the first order are likewise magnitudes of the first order, I call the multiplication an *outer* one (aeussere).

15. *If A, B, C, D, are points, then we mean by*

(1) $A\,.\,B$, *the line, which has A and B as extremities, regarded as a definite part of the infinite right line determined by A and B,*

(2) $A\,.\,B\,.\,C$, *the triangle, whose vertices are A, B, C, regarded as a definite part of the infinite plane determined by A, B, C,*

(3) $A\,.\,B\,.\,C\,.\,D$, *the tetraedron, whose vertices are A, B, C, D, regarded as a definite part of infinite space.*

That is, we put $A\,.\,B = A_1\,.\,B_1$, when both products represent equal p'ts, with like signs,* of the same right line; further

$$A\,.\,B\,.\,C = A_1\,.\,B_1\,.\,C_1,$$

when both triangles are equal parts, with like signs, of the same plane; and finally

$$A\,.\,B\,.\,C\,.\,D = A_1\,.\,B_1\,.\,C_1\,.\,D_1,$$

when both tetraedrons have equal volumes, with like signs.

16. *If* a, b, c, *are lines of definite length and direction, then we mean by*

(1) a . b, *the parallelogram, whose sides are equal and parallel to* a *and* b, *regarded as an area of definite magnitude and aspect of plane.*†

(2) a . b . c, *the parallelepiped, whose edges are equal and parallel to* a, b, c, *regarded as a volume of definite magnitude.* That is, we put

$$a\,.\,b = a_1\,.\,b_1,$$

when the parallelograms, which are represented by these products lie in parallel planes, and have the same areas with like signs;

$$a\,.\,b\,.\,c = a_1\,.\,b_1\,.\,c_1,$$

when the parallelepipeds represented by these products have equal volumes with like signs.

17. *The side* (*right or left*) *toward which a spatial magnitude is constructed, determines its positive or negative value.*

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*Two magnitudes have like signs, when they have either both a positive, or both a negative value.

†I say that two parallel planes have the same aspect.

(1) Two parts of the same line, $A . B$ and $A_1 . B_1$, we regard as having the same sign, when B lies on the same side of A, as B_1 of A_1.

(2) Two parts of the same plane $A . B . C$ and $A_1 . B_1 . C_1$, we regard as having the same sign, when C lies on the same side of $A . B$, as C_1 of $A_1 . B_1$; or, more plainly, when C is on the same side of one standing at A and looking toward B, as C_1 of one standing at A_1 and looking toward B_1.

(3) Two parts of the same solid, $A . B . C . D$, and $A_1 . B_1 . C_1 . D_1$, we regard as having the same sign, when D lies on the same side of $A . B . C$, as D_1 of $A_1 . B_1 . C_1$; or, more plainly, if the point D lies on the same side of a human body whose head is directed toward A, feet toward B, and eye toward C, as D_1 is of a body whose head is directed toward A_1, feet toward B_1, and eye toward C_1.

(4) Two parallel surfaces $a . b$ and $a_1 . b_1$ we regard as having the same sign when the direction b lies on the same side of direction a, as b_1 of a_1.

(5) Two solids $a . b . c$ and $a_1 . b_1 . c_1$ we regard as having the same sign when the direction c lies on the same side of $a . b$, as c_1 of $a_1 . b_1$, i. e. when the direction c lies on the same side of a human body in which the direction a leads from the feet to the head, and whose eyes look forward in the direction b, as the direction c_1 of a body, etc.

18. *There are seven classes of spatial magnitudes separated into four orders* (Stufen):

Ord. I. (1) *Simple or multiple points.*
(2) *Right lines of definite length and direction.*

Ord. II. (3) *Definite parts of given infinite right lines.*
(4) *Plane surfaces of definite magnitude and aspect of plane.*

Ord. III. (5) *Definite parts of given infinite planes.*
(6) *Definite volumes.*

Ord. IV. (7) *Definite volumes.*

Here volumes appear twice, once as magnitudes of the third order, again as magnitudes of the fourth order, according as they are regarded as the product of three right lines of definite direction and length, or as the product of four points.

19. *Parts, with like signs, of one and the same whole have as their sum a part, with the same sign, of the same whole, which part is as great as these two combined.*

For example, if $A . B$ and $A_1 . B_1$ are parts with the same direction, of the same infinite right line, they have as their sum, a part, with the same direction, of the same right line, which is as great as these two combined.

20. *Any two magnitudes of the same order, but only such, can be added; the meaning of the addition of such magnitudes can always be determined, if we adhere to the previously given signification of these magnitudes and apply the rules of III.*

Prob. 3. To add two points, A and B.

If we put $A+B=2S$, we get $B-A=2(S-A)$, i. e. S is the mean between A and B. Thus the sum of two points is twice the mean bet. them.

Prob. 4. To add two multiple points, αA and βB, when the coefficients, α and β, are positive, i. e. to find the point S, which satisfies the equation

$$\alpha A+\beta B = (\alpha+\beta)S.$$

If this equation is satisfied, we must have

$$\beta(B-A) = (\alpha+\beta)(S-A)$$

and, conversely, from the latter we get the former. But from the latter is obtained this construction:

Take on the line AB from A toward B the part $\beta\div(\alpha+\beta)$ [or from B toward A the part $\alpha\div(\alpha+\beta)$]; then the termination of this part is the p't S.—Hence "the sum of two multiple points with positive coefficients, is a point, multiplied by the sum of the coefficients, which so lies in the line between the two points that its distances from these two points are inversely proportional to the coefficients belonging to these points."*

Prob. 5. To add a point A and a right line of definite length and direction $C-B$.

Construct a right line from A equal in length to $C-B$ and having the same direction. Let this be $D-A$; then is D the sum required; for since $C-B=D-A$, we have

$$A+(C-B)=A+(D-A)=D.$$

Hence "the sum of a point A and a right line of definite length and direction is the termination of this line, when A is its origin."

Prob. 6. To add a multiple point αA and a right line of definite length and direction $C-B$.

Construct a right line from A having the same direction as $C-B$, but only $\frac{1}{\alpha}$th as long. Let this be $D-A$, then is αD the sum required. For since $C-B=\alpha(D-A)$, we have

$$\alpha A+(C-B)=\alpha A+\alpha(D-A)=\alpha D.$$

Prob. 7. To add two right lines of definite length and direction, $B-A$ and $D-C$.

Make $E-B = D-C$; then

$$(B-A)+(D-C)=(B-A)+(E-B)=E-A.$$

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*It is easily seen that this point is the center of gravity, when the coefficients represent weights.

Hence "two right lines of definite length and direction can be added by putting the origin of the second at the termination of the first, without changing length and direction, when the right line from the origin of the first to the termination of the second is the sum required."

Prob. 8. To add n right lines of definite length and direction.

The repeated application of the solution of Prob. 7 leads at once to the solution of this problem. Hence "n right lines of definite length and direction can be added by so joining the lines in a continuous series, without changing length and direction, that where one terminates, the next following begins; then is the right line from the origin of the first to the termination of the last the sum required."

Prob. 9. To find the sum of n points, $A_1, A_2, A_3, \dots A_n$, i. e. to find the point S, which satisfies the equation

$$A_1+A_2+A_3+ \dots A_n = S.$$

If we subtract from both members of the equation nR, where R is any point, we get

$$(A_1-R)+(A_2-R)+(A_3-R)+ \dots (A_n-R) = n(S-R).$$

Now, since the first equation may be deduced again from this, we have the following: "To add n points, draw from any point R right lines to the n points, join them continuously to one another without changing length and direction, beginning with the origin of the first at R, connect R with the termination of the last by a right line, and divide this right line into n equal parts; then is the first point of division from R the point S, n times which is the sum required."

Prob. 10. To add any number of multiple points, $\alpha A, \beta B, \gamma C \dots\dots$, when the sum of the coefficients $\alpha + \beta + \gamma + \dots$ is not zero.

Put $$\alpha A+\beta B+\gamma C+ \dots = (\alpha+\beta+\gamma \dots)S,$$
and subtract from both members $(\alpha + \beta + \gamma \dots)R$, where R is any point; then we get

$$\alpha(A-R)+\beta(B-R)+\gamma(C-R)+ \dots = (\alpha+\beta+\gamma+ \dots)(S - R).$$

Now, since the first equation may be deduced again from this, we have the following: "To find the sum of any number of multiple points αA, βB, γC - - -, the sum of whose coefficients is not zero, draw from any point R lines to A, B, C - - -, multiply these by α, β, γ - - -, respectively,* join the lines so obtained continuously to one another without changing direction and length, beginning with the origin of the first at R, connect R with the termination of the last by a right line, and take upon this line from R a distance equal to the $\frac{1}{\alpha+\beta+\gamma+\ldots}$th part; then is the termination of this distance multiplied by $(\alpha+\beta+\gamma+\ldots)$ the sum required.

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*In such multiplication by a numerical quantity α the direction is not changed when α is positive, as is easily seen, while the length is changed in the ratio $1:\alpha$; if α is negative the direction becomes the opposite.

Prob. 11. To find the sum of multiple points, αA, βB, . . . when $\alpha+\beta+\gamma+\ldots=0$.

Subtract from the sum $\alpha A+\beta B+\gamma C+\ldots$ the expression $(\alpha+\beta+\gamma+\ldots)R$; then since this subtracted quantity is zero, the value of the sum is not changed and we have

$$\alpha A+\beta B+\gamma C+\ldots=\alpha(A-R)+\beta(B-R)+\gamma(C-R)+\ldots$$

Hence "the sum of multiple points, the sum of whose coefficients is zero, is a right line of definite length and direction, which may be obtained by drawing from any point R right lines to the given points, multiplying them by the coefficients belonging to these points and adding the products."

Prob. 12. To add two parts $A\,.\,B$ and $C\,.\,D$ of lines which intersect in E."

Make $E\,.\,F=A\,.\,B$, and $E\,.\,G=C\,.\,D$; then $A\,.\,B+C\,.\,D=E\,.\,F+E\,.\,G=E\,.\,(F+G)=2E\,.\,S$, if S is the mean of F and G (Prob. 3).

Hence "To find the sum of two parts of lines which intersect; take the point of intersection of these lines as the origin; then twice the right line from the intersection to the middle point of the two extremities is the sum required."*

Prob. 13. To add two parts $A\,.\,B$ and $C\,.\,D$ of parallel lines, when they are of unequal length, and have opposite directions.

If $A\,.\,B$ and $C\,.\,D$ are parallel, $D-C$ must be equal to $\alpha(B-A)$, where α is any positive or negative number. Now, since $A\,.\,B$ is equal to $A\,.\,(B-A)$, because $A\,.\,A=0$ (No. 12), we have

$$\begin{aligned}A\,.\,B+C\,.\,D&=A\,.\,(B-A)+C\,.\,(D-C)\\&=A\,.\,(B-A)+\alpha C\,.\,(B-A)\\&=(A+\alpha C)(B-A).\end{aligned}$$

If the sum $A+\alpha C=(1+\alpha)S$ (see Prob. 4), then the last expression

$$=S\,.\,(1+\alpha)(B-A)=S\,.\,(B-A+D-C)$$

from which a simple construction of that sum may be obtained.

Prob. 14. To add two parts of lines $A\,.\,B$ and $C\,.\,D$, when they are of equal length and have opposite directions.

If both lie in the same right line, their sum is zero. If this is not the case, we have, since $D-C=-(B-A)$,

$$\begin{aligned}A\,.\,B+C\,.\,D&=A\,.\,(B-A)+C\,.\,(D-C)\\&=A\,.\,(B-A)-C\,.\,(B-A)\\&=(A-C)(B-A).\end{aligned}$$

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*This is the diag'l of the parallelogram which has these parts of lines as its sides, whence we see that the sum of the parts of lines is the resultant when the parts of lines repr't forces.

The sum therefore is an area of definite magnitude and aspect of plane.

Prob. 15. To add two surfaces, $a.b$ and $c.d$, of definite magnitude and aspect of plane.

If the planes are parallel, they can be added according to (No. 19); if they are not, both planes will have a direction in common. Let e be a right line having this direction, and $a.b=e.f$, $c.d=e.g$; then

$$a.b+c.d=e.f+e.g=e.(f+g).$$

Prob. 16. To add two parts $A.B.C$ and $D.E.F$ of definite planes which are not parallel.

If the planes are not parallel they will intersect. Let $G.H$ be a part of their line of intersection, and let $A.B.C=G.H.J$, $D.E.F=G.H.K$. Then

$$\begin{aligned}A.B.C+D.E.F&=G.H.J+G.H.K\\&=G.H.(J+K)=2G.H.S,\end{aligned}$$

if S be the mean between J and K. Hence "To add two parts of planes not parallel, represent them as triangles whose common base lies in the intersection of the two planes; then twice the triangle having the same base, whose vertex is the middle point between the vertices of these triangles is the sum required."

Prob. 17. To add two parts, $A.B.C$ and $D.E.F$, of parallel planes.

If the planes are parallel, we can make $(E-D).(F-D)=a.(B-A).(C-A)$, where a is a numerical quantity. Hence

$$\begin{aligned}A.B.C+D.E.F&=A.(B-A).(C-A)+D.(E-D).(F-D)\quad\text{(No. 12)}\\&=(A+aD).(B-A).(C-A)\\&=S.(1+a).(B-A).(C-A),\end{aligned}$$

if $(1+a)S$ is the sum $A+aD$. The last expression

$$=S[(B-A).(C-A)+(E-D).(F-D)],$$

in which a simple construction is again manifest. If, however, $a=-1$, i. e. if both figures are equal in area but have opposite signs, then $A+aD$ is a right line of definite direction and length (Prob. 11). Let this equal $H-G$. Then

$$A.B.C+D.E.F=(H-G).(B-A).(C-A),$$

and hence the sum is a volume.

Prob. 18. To add a part $A.B.C$ of a definite plane, and a volume $(D-A).(B-A).(C-A)$.

$$\begin{aligned}A.B.C+(D-A).(B-A).(C-A)&=A.(B-A).(C-A)+(D-A).(B-A).(C-A)\\&=D.(B-A).(C-A),\end{aligned}$$

from which the meaning of this addition is easily seen.

21. *A combinatory product whose factors of the first order* (Ordnung) *are magnitudes of the* $(n-1)$th *order* (Stufe), *but all of which lie in one and*

the same field of the n*th order, I call a regressive* (eingewandtes) *product taken with reference to that field,* e. g. a combinatory product of parts of lines in the plane, or of parts of planes in space.

22. If henceforth outer multiplication be designated by simply writing the factors together, regressive multiplication by a point placed between the factors, *we understand by the regressive product* $AB\,.\,AC$, *where* A, B, C, *are any magnitudes, the product* $ABC\,.\,A$, *in which* ABC *is treated as a coefficient belonging to* A, *provided that the product be referred to the field of lowest order in which* A, B *and* C *lie at the same time.*

Prob. 19. To find the product with reference to the plane ABC of two parts of lines $AB\,.\,AC$.

According to No. 22, this is equal to $ABC\,.\,A$; i. e. "the product of two parts of lines which intersect is their point of intersection combined with a part of the plane as a coefficient." If we regard a part of the plane as unity, the parts of planes by which the points are multiplied will be actual numerical quantities, and the products will appear as multiple points; all the magnitudes to be compared must then lie in the same plane, to which the products refer (as is always the case in plane geometry).

Prob. 20. To find the product of three segments of lines AB, AC, BC, with reference to the plane ABC.

Sol. $AB\,.\,AC\,.\,BC = ABC\,.\,ABC = (ABC)^2$.

Prob 21. To find the regressive product of two parts of planes ABC and ABD (with reference to the volume).

Sol. $ABC\,.\,ABD = ABCD\,.\,AB$.

Prob. 22. To find the regressive product of three parts of planes ABC, ABD, ACD.

Sol. $ABC\,.\,ABD\,.\,ACD = ABCD\,.\,ABCD\,.\,A = (ABCD)^2\,.\,A$.

Prob. 23. To find the regressive product of four parts of planes ABC, ABD, ACD, BCD.

Sol. $ABC.ABD\,.\,ACD\,.\,BCD = (ABCD)^3$.

Note. The product of two parts of planes is therefore a part of a line, of three a point, but the part of the line and the point still have a volume or a product of volumes as coefficients, and if we make a definite volume a unit, these coefficients become actual numerical quantities.

These are perhaps the most essential conceptions which appear in the first part of my extensive algebra. But it is impossible in this place to give any thing more than a superficial idea of the infinite fruitfulness of this new method for the treatment not only of geometry but, in general, of all sciences which are based upon spatial relations. As little could I in this place give the proofs that the rules of operation found in III. are applicable to the forms of combination here set forth, but here also I must refer to my extended treatise in which these proofs are drawn out with all requisite exactness; and where at the same time the development advances in such a way that every thing, which seems to be arbitrary in the presentation of the different ideas, vanishes.

GEOMETRICAL DETERMINATION OF THE SOLIDITY OF THE ELLIPSOID.

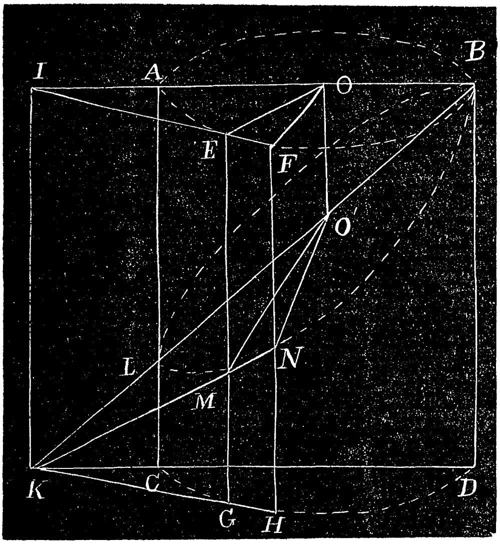
BY OCTAVIAN L. MATHIOT, BALTIMORE, MARYLND.

LET $ABCD$ be a right cylinder bounding a regular prism having an infinite number of sides, and let $EFGH$ represent one of these sides. Draw FEI tangent to the cylinder at F and meeting the diameter AB produced in I. From I let fall the perpendicular IK to meet the lower diameter CD produced in K. From K draw KGH a tangent to the cylind. at H.

From B to K pass a plane with cutting edge parallel to the lower base, and it will produce the ellipse $LMNB$ containing a polygon having MN for one of its infinite number of sides.

LB, the continuation of KL, will be the transverse axis of the ellipse, while the conjugate axis will be equal to the diameter AB of the cylinder.

From the centre O of the upper base let fall the perpendicular OO' and it will pass thro' the centre O' of the ellipse.



From O draw OE and OF, and from O', $O'M$ and $O'N$. From the similar triangles BOO' and BIK, we have $BO : BO' :: OI : O'K$. Since BO is half of AB and BO' is half of BL, BO = semi-conjugate diameter = B, and BO' = semi-transverse = A; $\therefore$ $OI : OK :: B : A$ But OI is the subtangent corresponding to the tangent line FEI of the circle, while $O'K$ is the subtangent crresponding to the tangent line KMN of the ellipse.